

# On the non-uniqueness problem of the covariant Dirac theory and the spin-rotation coupling

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## Abstract

The reason for the non-uniqueness problem is reexplained by using the notions of a unitary transformation and of the mean value of an operator, invoked by Gorbatenko & Neznamov [1]. Their arguments actually aim at proving the uniqueness of a particular prescription for solving this problem. But that prescription is again shown non-unique. Non-equivalent Hamiltonians in a rotating frame do ask physical questions about the spin-rotation coupling.

**Keywords:** Dirac Hamiltonian, curved spacetime, rotating frame

## 1 Introduction

In a most recent preprint [1], Gorbatenko & Neznamov write that “publications have emerged again [2, 3, 4], which declare and provide grounds for the assertion that the Dirac theory is non-unique in a curved and even flat spacetime”. They announce that, in contrast: “in this work we again prove the absence of the non-uniqueness problem of the Dirac theory in a curved and flat spacetime and illustrate this with a number of examples.” The aim of this paper is to show that they do not prove a such thing and that their examples do not and can not do that either. Their arguments do not address the former proof [2] of the generic non-uniqueness of the Hamiltonian and energy operators associated in a given reference frame with the (generally-)covariant Dirac equation—be it in a curved or in a flat spacetime, indeed. Nor do their arguments answer my former proof [3] that their algorithm based on going to a special kind of tetrad leaves the Hamiltonian and energy operators ambiguous. That proof used a counterexample that is relevant precisely to my discussion of the spin-rotation coupling [4], commented on in their Examples 6 and 7.

## 2 Physically equivalent operators

I fully agree with Gorbatenko & Neznamov that the mere “demonstration that the form of Dirac Hamiltonians depends on the choice of tetrads” would be “absolutely insufficient” to “demonstrate the non-equivalence of Dirac Hamiltonians”. However, in spite of what they state, this was not at all the approach followed in the paper in which, together with F. Reifler, we proved the non-uniqueness of the Hamiltonian and energy operators of the covariant Dirac theory [2]. On the contrary, in the paper [2], we began the study of that problem with carefully establishing the condition under which one may say that two versions of a quantum-mechanical operator such as the Hamiltonian, got by choosing two admissible coefficient fields in the covariant Dirac equation, are *physically equivalent*. Indeed the non-uniqueness problem is related, though not in a trivial way, with the fact that there is a vast continuum of different choices for the coefficient fields of the covariant Dirac equation. Any two such fields, thus any two fields of Dirac matrices  $\gamma^\mu$  and  $\tilde{\gamma}^\mu$ , are related together by a “local” similarity transformation, given by a non-singular complex matrix  $S(X)$  that depends smoothly on the point  $X$  in the spacetime  $V$ :<sup>1</sup>

$$\tilde{\gamma}^\mu = S^{-1} \gamma^\mu S, \quad \mu = 0, \dots, 3. \quad (1)$$

We noted first that, with each of the two different coefficient fields:  $\gamma^\mu$  and  $\tilde{\gamma}^\mu$ , corresponds a unique Hilbert scalar product. Explicitly:

$$(\Psi | \Phi) \equiv \int \Psi^\dagger \sqrt{-g} A \gamma^0 \Phi \, d^3 \mathbf{x} \quad (2)$$

for the first field,  $\gamma^\mu$ , and

$$(\Xi | \Omega) \equiv \int \Xi^\dagger \sqrt{-g} \tilde{A} \tilde{\gamma}^0 \Omega \, d^3 \mathbf{x}, \quad \tilde{A} \equiv S^\dagger A S \quad (3)$$

for the second one,  $\tilde{\gamma}^\mu$  (with  $S^\dagger A S = A$  if  $S$  is an admissible similarity transformation for DFW). Thus, with the first coefficient field  $\gamma^\mu$ , the wave function  $\Psi$  lives in a Hilbert space  $\mathcal{H}$  and, with the second coefficient field  $\tilde{\gamma}^\mu$ , the wave function  $\Xi$  lives in a different Hilbert space  $\tilde{\mathcal{H}}$ . Moreover, when applied precisely to the *wave function*,

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<sup>1</sup> In this paper we shall consider only the standard version of the covariant Dirac equation, or “Dirac-Fock-Weyl” equation (DFW equation for short). For DFW, the hermitizing matrix  $A$  is a constant matrix which is invariant under any admissible local similarity transformation  $S$ , i.e., any one got from “lifting” a local Lorentz transformation applied to the (orthonormal) tetrad field {Ref. [5], Eq. (104) and below}. Then the coefficient fields are indeed reduced to the field  $\gamma^\mu$  ( $\mu = 0, \dots, 3$ ). In standard practice (including in Ref. [1]), one has even  $A = \gamma^{\sharp 0}$ , where  $(\gamma^{\sharp \alpha})$  is some special set of Dirac matrices for the Minkowski spacetime in Cartesian coordinates.

the similarity transformation  $S$  defines a transformation  $\mathcal{U}$  from the first Hilbert space  $\mathcal{H}$  onto the second one,  $\tilde{\mathcal{H}}$ :

$$\mathcal{U}\Psi \equiv \tilde{\Psi} \equiv S^{-1}\Psi, \quad i.e., \quad (\mathcal{U}\Psi)(X) \equiv S(X)^{-1}\Psi(X). \quad (4)$$

This one-to-one mapping is a (linear) isometry [2], or in other words a *unitary transformation*  $\mathcal{U}$  of  $\mathcal{H}$  onto  $\tilde{\mathcal{H}}$ , for we have from Eqs. (1) to (4):

$$\forall \Psi, \Phi \in \mathcal{H}, \quad (\mathcal{U}\Psi | \mathcal{U}\Phi) \equiv (\tilde{\Psi} | \tilde{\Phi}) = (\Psi | \Phi). \quad (5)$$

Under this unitary transformation, any quantum-mechanical operator such as, for example, the Hamiltonian operator  $\mathsf{H}$ , defined on  $\mathcal{H}$ —or rather on a dense subspace  $\mathcal{D}$  of  $\mathcal{H}$ —is carried over to the pushforward operator  $\check{\mathsf{H}}$  under  $\mathcal{U}$ , which is an operator defined on  $\check{\mathcal{D}} \equiv \mathcal{U}(\mathcal{D})$ :

$$\check{\mathsf{H}} \equiv \mathcal{U}\mathsf{H}\mathcal{U}^{-1}, \quad (6)$$

that is from (4):

$$\forall \Xi \in \check{\mathcal{D}} \equiv \mathcal{U}(\mathcal{D}), \quad \check{\mathsf{H}}\Xi = S^{-1}\mathsf{H}S\Xi. \quad (7)$$

The pushforward operator  $\check{\mathsf{H}}$  is *physically equivalent to the starting operator*  $\mathsf{H}$  since, from its definition (6) and the unitarity of  $\mathcal{U}$  (5), all products  $(\Psi | \mathsf{H}\Phi)$ ,  $\Psi, \Phi \in \mathcal{D}$ , stay unchanged after the unitary transformation  $\mathcal{U}$ :

$$\forall \Psi, \Phi \in \mathcal{D}, \quad (\mathcal{U}\Psi | \check{\mathsf{H}}(\mathcal{U}\Phi)) = (\Psi | \mathsf{H}\Phi). \quad (8)$$

Note in particular that, for any state  $\Psi \in \mathcal{D}$ , the mean value of  $\mathsf{H}$  for this state,  $\langle \mathsf{H} \rangle \equiv (\Psi | \mathsf{H}\Psi)$ , is equal to the mean value of  $\check{\mathsf{H}}$  for the transformed state after the unitary transformation:  $\langle \check{\mathsf{H}} \rangle \equiv (\mathcal{U}\Psi | \check{\mathsf{H}}(\mathcal{U}\Psi)) = \langle \mathsf{H} \rangle$ . Because a sesquilinear form is determined by the corresponding quadratic form, this is *characteristic* of the pushforward operator: if an operator  $\mathcal{O}$  defined on  $\check{\mathcal{D}} \equiv \mathcal{U}(\mathcal{D})$  is such that

$$\forall \Psi \in \mathcal{D}, \quad (\mathcal{U}\Psi | \mathcal{O}(\mathcal{U}\Psi)) = (\Psi | \mathsf{H}\Psi), \quad (9)$$

then we have also (8) with  $\mathcal{O}$  in the place of  $\check{\mathsf{H}}$ , whence follows {Ref. [2], Note 6} that  $\mathcal{O} = \check{\mathsf{H}}$ .

But the new operator, here the new Hamiltonian  $\check{\mathsf{H}}$ , corresponding with the new coefficient field  $\tilde{\gamma}^\mu$  after the similarity transformation  $S$ , is defined in a different way than  $\check{\mathsf{H}}$ —although obviously it also acts on  $\check{\mathcal{D}} \equiv \mathcal{U}(\mathcal{D})$ . Namely, the new operator is defined as the starting one, but replacing the starting coefficient fields by the new ones. Since undoubtedly two physically equivalent operators must yield the same mean value for any two unitarily-equivalent states  $\Psi$  and  $\tilde{\Psi} \equiv \mathcal{U}\Psi \equiv S^{-1}\Psi$ , we thus may state from the foregoing that, *in order that the Hamiltonian operator  $\check{\mathsf{H}}$  after*

the similarity transformation  $S$  be physically equivalent to the initial one  $H$ , it is necessary and sufficient that we have  $\tilde{H} = \check{H}$ , that is  $\{[2], \text{Eq. (43)}\}$ :

$$\tilde{H} = S^{-1} H S. \quad (10)$$

It is clear from the derivation of this result that it applies exactly in the same way to any other quantum-mechanical operator, such as e.g. the energy operator defined in Eq. (16) below.

### 3 Non-uniqueness of the Hamiltonian

The Hamiltonian operator [in a given coordinate system  $(x^\mu)$ ] is defined by rewriting the Dirac equation in Schrödinger form. Thus:

$$i \frac{\partial \Psi}{\partial t} = H \Psi \quad (t \equiv \frac{x^0}{c}), \quad (11)$$

and

$$i \frac{\partial \Xi}{\partial t} = \tilde{H} \Xi, \quad (12)$$

respectively before and after application of a local similarity transformation  $S$ . As is also well known [6, 7], the DFW equation is covariant (in a topologically-simple spacetime) under those local similarity transformations that are admissible, i.e., those that are got by lifting a local Lorentz transformation applied to the tetrad field. This means that, for a such admissible similarity transformation  $S$ , Eqs. (11) and (12) are equivalent if one exchanges the wave functions  $\Psi$  and  $\Xi$  according to the unitary transformation (4):

$$\Xi = \mathcal{U} \Psi \equiv S^{-1} \Psi. \quad (13)$$

Substituting thus  $\Psi = S \Xi$  into (11) and using (12), we get:

$$\tilde{H} = S^{-1} H S - i S^{-1} \partial_t S = \check{H} - i S^{-1} \partial_t S. \quad (14)$$

Comparing with (10), we recover in a simple way [3] the result  $\{[2], \text{Eq. (48)}\}$  that: *For DFW, in order that the Hamiltonian operator  $\tilde{H}$  after the similarity transformation  $S$  be physically equivalent to the initial one  $H$ , it is necessary and sufficient that the similarity  $S$  be independent of the time  $t$ :*

$$\partial_t S = 0. \quad (15)$$

(In Ref. [2], this had been got by comparing the explicit expressions of  $\tilde{H}$  and  $H$ .) *This result* and a similar result that gives the condition in order that the energy

operators before and after the similarity be equivalent, *not* the “demonstration that the form of Dirac Hamiltonians depends on the choice of tetrads”, was the basis for our statement of the generic non-uniqueness of the DFW Hamiltonian and energy operators. Indeed, nothing prevents one from changing the coefficients by a *time-dependent* similarity, which leads hence to inequivalent Hamiltonians [2].

**In summary:** a change of the tetrad field defines an admissible similarity transformation  $S$  that applies both to the field of Dirac matrices by (1), and to the wave function by (4). The transformation (4) is in fact a unitary transformation  $\mathcal{U}$  between the Hilbert spaces in which the wave function lives before and after the similarity transformation  $S$ , Eq. (5). However, when  $S$  depends on the time coordinate  $t$ , the new Hamiltonian operator  $\tilde{H}$ , after the similarity, does not coincide with the pushforward operator  $\tilde{H}$  of the initial Hamiltonian  $H$  under the unitary transformation  $\mathcal{U}$ , Eq. (14). Hence, when  $S$  depends on  $t$ , the mean values of  $H$  and  $\tilde{H}$  cannot coincide for all states, so that  $H$  and  $\tilde{H}$  are not physically equivalent.

The origin of the non-uniqueness problem is thus not trivial. It does not reside in the mere fact that many different fields of Dirac matrices can indifferently be chosen, nor in the other obvious fact that these different fields lead in general to different forms of the Hamiltonian operator in a given coordinate system. As I just showed anew, the non-uniqueness problem applies to the Hamiltonian associated with the standard version of the covariant Dirac equation, “DFW” (as well as to the Hamiltonian associated with alternative versions of the covariant Dirac equation [2]), although the DFW equation itself has been carefully built so as to be covariant under the admissible similarity transformations, thus essentially unique. Only the latter uniqueness (the covariance of the DFW equation under the admissible similarity transformations) was included in the “conclusions of previous studies [6, 7] on the independence of physical characteristics of the Dirac theory on the choice of tetrads”, which Gorbatenko & Neznamov [1] say to share. The non-uniqueness applies also [2] to the *energy operator*  $E$ , that coincides with the Hermitian part, for the scalar product (2), of the Hamiltonian operator:

$$E = \frac{1}{2}(H + H^\dagger). \quad (16)$$

The non-uniqueness applies also to the *spectrum* of the energy operator in a given coordinate system, as was proved in Ref. [2], and this is also true in the presence of an electromagnetic field [13].

## 4 Trying to solve the non-uniqueness problem

In a general coordinate system in a general spacetime, the metric depends on the time coordinate  $t$ , and then so does the field of orthonormal tetrads defining the field of Dirac matrices. It is then the general case that the local Lorentz transformation  $L$  relating two different tetrad fields depend on  $t$ , so that the similarity transformation  $S$  got by “lifting”  $L$  also depend on  $t$ , thus leading to a Hamiltonian  $\tilde{H}$  that is not equivalent to the starting one  $H$ , see Eq. (14). This does not mean, of course, that *all* pairs  $(H, \tilde{H})$ , got by choosing one admissible tetrad field and transforming it to a new tetrad field through a local Lorentz transformation, are made of two inequivalent operators. *Therefore, exhibiting some pairs  $(H, \tilde{H})$  that are (supposedly) made of two equivalent operators, as do Gorbatenko & Neznamov [1], can not disprove the existence of the non-uniqueness problem.* In particular, in a coordinate system in which the metric is stationary:  $g_{\mu\nu,0} = 0$ , it is natural (though, of course, not mandatory) to choose time-independent tetrads, thus leading to coefficient fields related two by two by a time-independent similarity transformation, hence giving equivalent Hamiltonian operators. Such is the case for several among the examples given in Ref. [1]: most certainly for Example 4 based on the static, diagonal, space-isotropic metric considered among others by Obukhov [8] and by Silenko & Teryaev [9], but likely also for some of the examples based on the Kerr metric, which is stationary.

Once the generic non-uniqueness of the Dirac Hamiltonian and energy operators is recognized, to escape this non-uniqueness demands to build some *prescription* that restrict the choice of tetrad when the coordinate system and the corresponding expression of the metric are given. That prescription should be consistent (i.e., well defined), moreover the corresponding restriction in the choice of the tetrad field should be sufficient (i.e., it should solve the problem), and preferably it should be physically motivated. Building a such prescription necessarily involves some choice, which is not strongly constrained in the current state of experimental knowledge. (Nevertheless, choosing tetrad fields with high rotation rates, without any relation to the rotation of a physical body, would lead to high theory-experiment discrepancies. Also recall that the non-uniqueness problem is already there in a flat spacetime as soon as one uses the DFW equation with its gauge freedom, and this also in the presence of the electromagnetic field, so that even the energy levels of the hydrogen atom would not be defined [13].) However, just finding one definite prescription (even an artificial, computationally-motivated one) that *really* provide unique Hamiltonian and energy operators, is difficult.

In a series of papers by Gorbatenko & Neznamov [10, 11, 12], attempts have been made at finding a such prescription.<sup>2</sup> Their first paper [10] was limited to

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<sup>2</sup> The very title of the paper [10] indicates clearly that its authors aimed at solving some non-

a time-independent metric. For that case, as recalled above, one might content oneself with choosing time-independent tetrad fields, since any two of them lead to equivalent Hamiltonian and to equivalent energy operators [2]. Their proposal for the general, time-dependent case [11, 12] was discussed in detail in Ref. [3]. As noted there, that proposal consists first in going from the starting arbitrary [orthonormal] tetrad field, say  $(u_\alpha)$ , to what the authors name “the Schwinger tetrad”, and which is a tetrad field, say  $(\tilde{u}_\alpha)$ , of which each among the three “spatial” vector (fields)  $u_p$  ( $p = 1, 2, 3$ ) has a zero “time” component, in the coordinate system considered. Their procedure based on the formalism of pseudo-Hermitian Hamiltonians leads them then to define as the candidate for a unique Hamiltonian the one noted  $H_\eta$ , got after a similarity transformation denoted  $\eta$  by them, and which in the form (1) I will denote by  $T = \eta^{-1}$ . We have {Ref. [11], Eq. (66)}:

$$T = a^{-1}S \equiv (\eta_{\text{G\&N}})^{-1}, \quad (17)$$

where  $S$  is the admissible similarity transformation associated with the change of tetrad from  $(u_\alpha)$  to  $(\tilde{u}_\alpha)$ , and  $a = |g g^{00}|^{1/4}$ . ( $S^{-1}$  is what these authors note  $L$ .) It is easy to see, as they also note [12], that the Hamiltonian  $H_\eta$  got after the similarity transformation  $T$  is *also* equal to the *energy operator* [Eq. (16) in the present paper] got with the field of Dirac matrices deduced from the starting field by the admissible similarity transformation  $S$ . Note that their procedure is really a *prescription* for restricting the choice of tetrad field in order to (try to) get unique Hamiltonian and energy operators, of the general kind I described in the foregoing paragraph. Indeed, not every tetrad field is a “Schwinger tetrad” in a given coordinate system. Thus, *if their procedure would actually lead to a unique Hermitian Hamiltonian in a given coordinate system, that would not prove anything against the existence of the non-uniqueness problem*: that would be a first check of a prescription for solving it.

<sup>3</sup> But that is not the case, as I will now show.

## 5 The “Schwinger tetrad” prescription is not unique

As I showed in detail in Ref. [3], App. C, the choice of a tetrad that, in a given coordinate system, is a “Schwinger tetrad”, is far from unique. Gorbatenko & Neznamov did not prove that, if in the same coordinate system one takes a second Schwinger tetrad, then the Hermitian Hamiltonian provided by their construction from that

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uniqueness problem for the Dirac Hamiltonian in a curved spacetime. To the best of my knowledge, the existence of a such problem has been noted for the first time in Ref. [5], and shown in detail in Ref. [2].

<sup>3</sup> As shown in Ref. [13], Sect. 4, it is actually not enough to get unique Hamiltonian and energy operators in any given coordinate system, for what is physically given is the reference frame (a three-dimensional congruence of time-like world lines), not the coordinate system, for which there is a vast functional space of different choices within a given reference frame.

second Schwinger tetrad is physically equivalent to the Hermitian Hamiltonian got from the first Schwinger tetrad, *and indeed that is not generally the case*. In Ref. [3], App. C, I made remarks that indicated this fact, and I gave a precise counterexample to illustrate this. Now I will prove a general result.

After the similarity transformation (17), the scalar product becomes the “flat” one, i.e.,  $\sqrt{-g} \tilde{A} \tilde{\gamma}^0 = \mathbf{1}_4$  in Eq. (3), as stated by Gorbatenko & Neznamov. This results from Eq. (65) in Ref. [11], and from the fact that, after a general similarity transformation  $T$ , the matrix  $M \equiv \sqrt{-g} A \gamma^0$  transforms like this [see Eqs. (1) and (3)<sub>2</sub>]:

$$\tilde{M} \equiv \sqrt{-g} \tilde{A} \tilde{\gamma}^0 = T^\dagger M T. \quad (18)$$

Now, besides the first “Schwinger tetrad” ( $\tilde{u}_\alpha$ ), consider another one, ( $\check{u}_\alpha$ ), and let  $S'$  be the admissible similarity transformation associated with the change of tetrad from the starting arbitrary tetrad ( $u_\alpha$ ) to ( $\check{u}_\alpha$ ). As in Eq. (17), let  $T' = a^{-1} S' = \eta'^{-1}$  be the similarity transformation leading to the other candidate Hamiltonian  $H_{\eta'}$ , corresponding to this other choice of a Schwinger tetrad. The change from the Hamiltonian  $H_\eta$  to the Hamiltonian  $H_{\eta'}$  is through the similarity

$$U = T' T^{-1} = S' S^{-1}, \quad (19)$$

which is admissible as are  $S$  and  $S'$ . Since the matrix  $M$  is equal to  $\mathbf{1}_4$  before and after the application of  $U$ , it results from the transformation law (18) that  $U = U(X)$  is a unitary *matrix*,  $U^\dagger U = \mathbf{1}_4$ . The operators  $H_\eta$  and  $H_{\eta'}$  can be said physically equivalent iff Eq. (10) is verified, which writes here:

$$H_{\eta'} = U^{-1} H_\eta U. \quad (20)$$

Let us determine when this is true. Since  $H_\eta$  (respectively  $H_{\eta'}$ ) is equal to the energy operator got with the tetrad ( $\tilde{u}_\alpha$ ) (respectively with the tetrad ( $\check{u}_\alpha$ )), these two operators exchange by the admissible similarity  $S' S^{-1} = U$ . The condition in order that two DFW energy operators related by an admissible similarity transformation  $U$  be equivalent, Eq. (20), is given by Eq. (64) in Ref. [2]:

$$B(\partial_t U) U^{-1} - [B(\partial_t U) U^{-1}]^\dagger = 0, \quad B \equiv A \gamma^0, \quad (21)$$

with  $\gamma^\mu$  the field of Dirac matrices for the first energy operator. When the first tetrad, corresponding with that field  $\gamma^\mu$ , is a “Schwinger tetrad”, we have  $B = \sqrt{g^{00}} \mathbf{1}_4$  {Ref. [2], Eq. (78). We assume here that  $A = \gamma^{b0}$ , as is standard: see Note 1.} Thus the condition in order that we have (20) is simply:

$$(\partial_t U) U^{-1} = [(\partial_t U) U^{-1}]^\dagger. \quad (22)$$



But since here  $U$  is a unitary matrix, it is immediate to check that this is true if and only if  $\partial_t U = 0$ . We have proved the following:

(i) *The energy operators  $H_\eta$  and  $H_{\eta'}$  corresponding with two different tetrads, each of which is a Schwinger tetrad in the same given coordinate system, exchange by an admissible similarity transformation whose matrix  $U(X)$  is unitary.*

(ii) *In order that  $H_\eta$  and  $H_{\eta'}$  be physically equivalent, it is necessary and sufficient that  $\partial_t U = 0$ .*

However, once again, it turns out to be the general case that  $U$  depend on  $t \equiv x^0/c$ . Indeed, given that  $(\tilde{u}_\alpha)$  is a Schwinger tetrad in some coordinate system  $(x^\mu)$ , and given a local Lorentz transformation  $L$ , the necessary and sufficient condition in order that the tetrad  $\check{u}_\beta = L^\alpha_\beta \tilde{u}_\alpha$  be also a Schwinger tetrad in the same coordinate system is that {Ref. [3], Eq. (89)}:

$$L^0_p = 0 \quad (p = 1, 2, 3). \quad (23)$$

In general, a local Lorentz transformation  $L$  verifying this condition depends on  $t$ , and so does the associated admissible similarity transformation  $U$ , got from  $L$  by using the spinor representation  $S$  (defined up to a sign):  $U = \pm S(L)$ .

## 6 The spin-rotation coupling

In their Examples 6 and 7, Gorbatenko & Neznamov [1] comment on my discussion (Ref. [4], Sect. 4) of the DFW Hamiltonians got in two reference frames in a Minkowski spacetime, when using three different tetrad fields. Their comment is limited to two tetrad fields: i)  $u'_\alpha \equiv \delta^\mu_\alpha \partial'_\mu$ , that is, the natural basis  $(\partial'_\mu)$  of a Cartesian coordinate system  $(x'^\mu) = (ct', x', y', z')$ , or “Cartesian tetrad”; and ii) the tetrad  $(u_\alpha)$  got from  $(u'_\alpha)$  by a spatial rotation of angle  $\omega t$  around the axis  $Oz'$ , with  $\omega$  a real constant {Ref. [4], Eqs. (34)–(35)}. Using Eqs. (33) and (35) of Ref. [4], one checks immediately that both tetrads  $(u'_\alpha)$  and  $(u_\alpha)$  are Schwinger tetrads in the Cartesian coordinates, as well as in the rotating coordinates  $(x^\mu) = (ct, x, y, z)$  given by

$$t = t', \quad x = x' \cos \omega t + y' \sin \omega t, \quad y = -x' \sin \omega t + y' \cos \omega t, \quad z = z'. \quad (24)$$

Therefore, the discussion in Sect. 5 applies. Moreover, as I noted [4], the Hamiltonians with the Cartesian tetrad  $(u'_\alpha)$ :  $H'_1$  in the inertial frame and  $H_1$  in the rotating frame, are Hermitian, as are also those with the rotating tetrad  $(u_\alpha)$ :  $H'_3$  in the inertial frame and  $H_3$  in the rotating frame. Thus each among these Hamiltonians

coincides with the corresponding energy operator.

Their Example 6 comments on the Hamiltonians in the inertial reference frame,  $H'_1$  and  $H'_3$  {respectively Eqs. (26) and (66) in Ref. [4]}, which are rewritten by the authors of Ref. [1] as their Eqs. (28) and (30), respectively. As I noted already in Ref. [3] [Eq. (94) there], the two tetrads  $(u'_\alpha)$  and  $(u_\alpha)$  exchange by a time-dependent Lorentz transformation  $L = L(t)$ : namely, the rotation of angle  $\omega t$  around the  $Oz'$  axis. Hence, the corresponding Hamiltonians  $H'_1 = H'_{G\&N}$  and  $H'_3 = H_{G\&N}$  exchange by the time-dependent similarity transformation  $U(t) = \pm S(L(t))$ , *and thus are not physically equivalent* [3], *contrary to what Gorbatenko & Neznamov [1] state*. This is just confirmed by their Eqs. (32) and (33): the similarity matrix  $U(t)$  is what these authors note  $R^{-1} = R^\dagger$ , it is indeed a unitary *matrix* as proved generally at Point (i) in Sect. 5—but as proved at Point (ii) the energy operators  $H_\eta \equiv H'_1 = H'_{G\&N}$  and  $H_{\eta'} \equiv H'_3 = H_{G\&N}$  are *not* physically equivalent. Note that, with  $R = U^\dagger = U^{-1}$  (and  $S \equiv U$  here), their Eq. (33):

$$H = RH'R^\dagger - iR\frac{\partial R^\dagger}{\partial t} \quad (25)$$

is exactly the same as Eq. (14) above. This is of course expected. The first term on the r.h.s. is exactly equivalent to  $H'$ , see Eq. (10). From the explicit expression  $R = e^{\omega t N}$  with  $N$  a constant matrix:  $N \equiv (\alpha'^1 \alpha'^2)/2$  [1] (with  $N^\dagger = -N$ ), it follows for the additional term making  $H$  inequivalent to  $H'$ :  $R\frac{\partial R^\dagger}{\partial t} = -e^{\omega t N} \omega N e^{-\omega t N} = -\omega N$ , which can be made *arbitrarily large* by taking a large number  $\omega$ .

Their Example 7 comments on the Hamiltonians in the uniformly rotating reference frame,  $H_1$  and  $H_3$  {respectively Eqs. (32) and (70) in Ref. [4]}, which are rewritten by the authors of Ref. [1] as their Eqs. (38) and (40), respectively. Exactly the same can be written as for Example 6, because the two tetrads and hence the similarity  $U$  are the same in the two examples, and, since  $U = U(t)$ , we have  $\partial_t U = \partial_{t'} U$  from Eq. (24) above.

Now we have the fact [4] that the spin-rotation coupling term  $-\frac{\hbar\omega}{2}\Sigma^3 = -\boldsymbol{\omega}\cdot\mathbf{S}$  [14, 15] is indeed involved in one among two Hamiltonians/energy operators in the uniformly rotating reference frame:  $H_3$ , but not in the other one:  $H_1$  (see also Ryder [16]). This fact cannot be discarded by stating that “the spin-rotation coupling has no effect on the final physical characteristics of the quantum mechanical systems under consideration” [1], because the two energy operators  $H_3$  and  $H_1$  are not physically equivalent. Nor, for the same reason, can the surprising fact [4] that the Hamiltonian/energy operator  $H'_3$  in the *inertial frame* does have the spin-rotation coupling term be discarded. But these two facts should lead one to ask whether this term must be there or not. In the Conclusion of Ref. [4], I explain why I believe

that the answer has to be experimental: it amounts to empirically deciding between two non-equivalent prescriptions [3, 13] for solving the non-uniqueness problem.

## 7 Summary

The reason for the non-uniqueness problem [2] has been reexplained and summarized by appealing precisely to the notions of a unitary transformation and of the mean value of an operator, invoked by Gorbatenko & Neznamov [1]. Their arguments actually aim at proving, not the uniqueness of the covariant Dirac theory, but the uniqueness that would be got (in their opinion) by using their particular prescription [11, 12] to select the tetrad field. Although I showed this already by exhibiting a counterexample [3], I showed here in a more general way that their prescription does not solve the non-uniqueness problem. Finally, the non-uniqueness of the Hamiltonian cannot be disproved by exhibiting some pairs of equivalent Hamiltonians. However, the examples regarding my discussion [4] of the spin-rotation coupling are made of pairs of grossly non-equivalent Hamiltonians.

## References

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